

# QUONS AS $su(2)$ IRREDUCIBLE TENSOR OPERATORS

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## Abstract

We prove that, for the quon algebra, which interpolates between the Bose and Fermi statistics and depends on a free parameter  $q$ , it is possible to build an  $su(2)$  irreducible representation. One of the consequences of this fact is that the quons couple via the same angular momentum coupling rules obeyed by ordinary bosons and fermions.

# 1 Introduction

The *quons* are particles whose statistics interpolates between the boson and fermion statistics and depend on a special deformation parameter which varies from  $+1$  to  $-1$  [1]. The usual bosonic (fermionic) algebra is recovered when the deformation parameter is equal to  $+1(-1)$ . The  $q$ -deformed commutation relation obeyed by the *quons* comes out as a specific case of the commutation relations obeyed by the  $q$ -deformed boson operators [2] of the quantum algebras [3] when we consider just one degree of freedom for the system. In general, however, they lead to different commutation relations and very specific properties, as discussed, for example, in reference [4]. The quon algebra comes out naturally in the study of the interactions of particles having small violations of Fermi and Bose statistics. Limits for these violations [1, ?] have been obtained in atomic and nuclear physics experiments.

In addition to these speculative works, the quon algebra has been employed as a tool to describe many-body problems involving composite particles [6]. In these problems, due to the simultaneous presence of composite and constituent degrees of freedom, the creation and annihilation operators of the composite particles deviates from the canonical ones. The deformation parameter  $q$  of the quon algebra may be considered as a measure of the effects of the statistics of the internal degrees of freedom of the composite particles.

Besides the intrinsic interest in the study of *quons* and of the fundamental behaviour of systems of particles that obey its statistics [5], the possibility of describing particles with properties which interpolate between fermion and boson ones is quite appealing in several areas of physics, specially in the treatment of many body systems through boson expansions [7],[8]. In the above mentioned works, as in many other possible applications, we have to deal with tensor operators that have to be coupled in order to define the physical operators of interest. In quantum algebras, it has been shown that  $q$ -deformed Clebsch-Gordan coefficients must be introduced, which are now available in the literature (see [9], for example). However they are not suitable for *quons*, once as we said before, the commutation relations and therefore the algebraic

structure obeyed by quons are different in general from the ones that define the quantum algebras.

In the present work we construct *quon* operators which behave as usual  $su(2)$  irreducible tensor operators. For this purpose we find a realization of the  $su(2)$  algebra in terms of the  $q$ -deformed bosons by building the  $su(2)$  generators  $J_+$ ,  $J_-$  and  $J_z$  [10] as functions of the quon operators. Within the quon algebra the number operators are written as series in terms of the deformation parameters. The  $su(2)$  generators built in this way also commute with the operators used to construct the hamiltonian.

## 2 The quon algebra

We assume that  $b_m, b_m^\dagger$ ,  $m = -j, \dots, +j$  are  $(2j+1)$  operators which satisfy the  $q$ -mutation relations (or the quon algebra) defined by [1], i.e.,

$$[b_m, b_{m'}^\dagger]_q = b_m b_{m'}^\dagger - q b_{m'}^\dagger b_m = \delta_{m,m'}. \quad (1)$$

This relation is a deformation of the Bose and Fermi algebras and interpolates between those algebras when  $q$  goes from  $+1$  to  $-1$ . The Fock space is constructed as usual from the application of the creation operators,  $(b_\alpha^\dagger)$ , on the vacuum state, defined as  $b_\alpha|0\rangle = 0$  for all  $\alpha$ . It has been shown[12, 11] that the squared norm of any polynomials of the creation operators,  $b_\alpha^\dagger$ , acting on the vacuum state is positive definite.

In what follows we derive some important results in order to construct the  $su(2)$  generators. We have at our disposal the transition number operator [1]  $N_{\alpha\beta}$ , which has the usual commutation relations:

$$[N_{\alpha\beta}, b_\mu^\dagger] = \delta_{\beta\mu} b_\alpha^\dagger, \quad [N_{\alpha\beta}, b_\mu] = -\delta_{\alpha\mu} b_\beta. \quad (2)$$

The transition number operator is an infinite series in the  $b_\mu^\dagger$ 's and  $b_\mu$ 's and it has been obtained in closed form in [11, 13]. The general structure of this

operator is [13] :

$$N_{\alpha\beta} = b_{\alpha}^{\dagger} b_{\beta} + \sum_{n=1}^{\infty} \sum_{(i_1 \dots i_n)} \sum_{\pi} c_{\pi(i_1), \dots, \pi(i_n), i_1 \dots i_n} (Y_{\alpha\pi(i_1), \dots, \pi(i_n)})^{\dagger} Y_{\beta i_1 \dots i_n} \quad (3)$$

where the summation over  $\pi$  means that we consider all different permutations of the indices  $i_1, i_2, \dots, i_n$ , including their repetitions.  $\pi(i_1), \pi(i_2), \dots, \pi(i_n)$  corresponds to each of these permutations and  $Y^{\dagger}$  is the adjoint of  $Y$ . The operators  $Y_{\beta i_1 \dots i_n}$  are obtained through the recursion relations [13, 14]:

$$Y_{k i_1 \dots i_{n+1}} = Y_{k i_1 \dots i_n} b_{i_{n+1}} - q^{n+1} b_{i_{n+1}} Y_{k i_1 \dots i_n} \quad (4)$$

with

$$Y_{ki} = b_k b_i - q b_i b_k$$

The coefficients  $c_{\pi(i_1), \dots, \pi(i_n), i_1 \dots i_n}$  are only functions of the deformation parameter  $q$  [13, 14]. The first terms in the series are:

$$N_{\alpha\beta} = b_{\alpha}^{\dagger} b_{\beta} + (1 - q^2)^{-1} \sum_m (b_m^{\dagger} b_{\alpha}^{\dagger} - q b_{\alpha}^{\dagger} b_m^{\dagger}) (b_{\beta} b_m - q b_m b_{\beta}) + \dots \quad (5)$$

From the recursion relation, eq.(4), and eq.(2), one can show that

$$\begin{aligned} [N_{\alpha\beta}, (Y_{\alpha' i_1 \dots i_n})^{\dagger}] &= \delta_{\beta i_1} (Y_{\alpha' \alpha i_2 \dots i_n})^{\dagger} + \delta_{\beta i_2} (Y_{\alpha' i_1 \alpha i_3 \dots i_n})^{\dagger} + \dots + \delta_{\beta i_{n-1}} (Y_{\alpha' i_1 i_2 \dots \alpha i_n})^{\dagger} \\ &+ \delta_{\beta i_n} (Y_{\alpha' i_1 i_2 \dots i_{n-1} \alpha})^{\dagger} + \delta_{\beta \alpha'} (Y_{\alpha i_1 i_2 \dots i_n})^{\dagger}. \end{aligned} \quad (6)$$

As a subsidiary result, it follows from eq.(6) that;

$$\begin{aligned} [N_{\alpha\beta}, (Y_{\alpha' \pi(i_1) \dots \pi(i_n)})^{\dagger} Y_{\beta' i_1 \dots i_n}] &= \delta_{\beta \alpha'} (Y_{\alpha \pi(i_1) \dots \pi(i_n)})^{\dagger} Y_{\beta' i_1 \dots i_n} \\ &- \delta_{\alpha \beta'} (Y_{\alpha' \pi i_1 \dots \pi i_{n+1}})^{\dagger} Y_{\beta i_1 \dots i_{n+1}}. \end{aligned} \quad (7)$$

In what follows we calculate the commutator  $[N_{\alpha\beta}, N_{\alpha'\beta'}]$ . For this purpose, we substitute  $N_{\alpha'\beta'}$  given in eq.(3) into the commutator above, which yields

$$\begin{aligned} [N_{\alpha\beta}, N_{\alpha'\beta'}] &= [N_{\alpha\beta}, b_{\alpha'}^{\dagger} b_{\beta'}] \\ &+ \sum_{n=1}^{\infty} \sum_{(i_1 \dots i_n)} \sum_{\pi} c_{\alpha' \pi(i_1), \dots, \pi(i_n), \beta' i_1 \dots i_n} [N_{\alpha\beta}, (Y_{\alpha' \pi(i_1), \dots, \pi(i_n)})^{\dagger} Y_{\beta' i_1 \dots i_n}]. \end{aligned}$$

Using eqs.(2) and (7), we obtain an important relation obeyed by the transition number operator:

$$[N_{\alpha\beta}, N_{\alpha'\beta'}] = \delta_{\beta\alpha'} N_{\alpha\beta'} - \delta_{\alpha\beta'} N_{\alpha'\beta} \quad . \quad (8)$$

The above commutation relation shows that the transition number operators are the generators of an  $su(2j+1)$  algebra.

### 3 $su(2)$ tensor operators

In order to see that the  $b_\mu^\dagger$  is an irreducible tensor operator, we have to show that

$$\begin{aligned} [J_0, b_\mu^\dagger] &= \mu b_\mu^\dagger \quad , \\ [J_\pm, b_\mu^\dagger] &= \sqrt{(j \mp \mu)(j \pm \mu + 1)} b_{\mu \pm 1}^\dagger \quad , \end{aligned} \quad (9)$$

where  $J_0, J_\pm$  are usual  $su(2)$  generators,

$$\begin{aligned} [J_0, J_\pm] &= \pm J_\pm \quad , \\ [J_+, J_-] &= 2J_0 \quad , \end{aligned} \quad (10)$$

which have to be built from the same  $b_\mu^\dagger$  operators. Notice that the label  $m$  corresponds to the projection of  $j$  and we are restricted to a single  $j$ -level. Operators belonging to different levels commute with each other. Thus we may define the  $su(2)$  generators as:

$$\begin{aligned} J_0 &= \sum_{\nu=-j}^{+j} \nu N_{\nu\nu} \\ J_\pm &= \sum_{\nu=-j}^{+j} \sqrt{(j \mp \nu)(j \pm \nu + 1)} N_{\nu \pm 1 \quad \nu} \end{aligned} \quad (11)$$

where the index  $\nu$  takes the values  $-j, -j+1, \dots, j-1, j$ . It is then simple to verify that the  $b_\mu^\dagger$  operators satisfy eqs.(9) and then form a genuine irreducible tensor operator of the  $su(2)$  algebra:

$$[J_0, b_\mu^\dagger] = \sum_\nu \nu [N_{\nu\nu}, b_\mu^\dagger] = \sum_\nu \delta_{\nu\mu} b_\mu^\dagger = \mu b_\mu^\dagger$$

and

$$[J_{\pm}, b_{\mu}^{\dagger}] = \sum_{\nu} \sqrt{(j \mp \nu)(j \pm \nu + 1)} [N_{\nu \pm 1 \ \nu}, b_{\mu}^{\dagger}] = \sum_{\nu} \sqrt{(j \mp \nu)(j \pm \nu + 1)} \delta_{\nu \mu} b_{\nu \pm 1}^{\dagger} =$$

$$\sqrt{(j \mp \mu)(j \pm \mu + 1)} b_{\mu \pm 1}^{\dagger} \quad .$$

To finish our demonstration we still have to show that the operators defined in eqs.(11), are indeed generators of the usual  $su(2)$  algebra. This proves the consistency of our approach. From eq.(11), we can write

$$[J_0, J_{\pm}] = \sum_{\nu, \nu'} \nu \sqrt{(j \mp \nu')(j \pm \nu' + 1)} [N_{\nu \nu}, N_{\nu' \pm 1 \ \nu'}] \quad (12)$$

$$= \sum_{\nu \ \nu'} \nu \sqrt{(j \mp \nu')(j \pm \nu' + 1)} (\delta_{\nu \ \nu' \pm 1} N_{\nu \nu'} - \delta_{\nu \ \nu'} N_{\nu' \pm 1 \ \nu}) = \pm J_{\pm} \quad .$$

One can show in an analogous way that  $[J_+, J_-] = 2 J_0$  and therefore the usual  $su(2)$  generators are consistent with the quon algebra, eq.(1).

## 4 Concluding Remarks

Thus, the tensor coupling has to be done with the usual Clebsch-Gordan coefficients.

In the Introduction we have mentioned, as an application of the *quon* statistics, the description of many body states by means of bosons, as it is done, for example in the IBM [15]. In that model the so called *s* and *d* bosons are introduced and in more sophisticated versions higher angular momentum bosons are also needed. It follows from the above demonstration that we can now define deformed *s* and *d* bosons, which behave like *quons* but keep the same angular momentum coupling rules and coefficients as in the non-deformed case. Of course, many other applications are possible.

This work has been partially supported by CNPq.

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